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An Improper Fractional Integral Involving Fractional Exponential Function

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Abstract: **In this paper, based on Jumarie type of Riemann-Liouville (R-L) fractional calculus and a new multiplication of fractional analytic functions, we use integration by parts for fractional calculus to solve an improper fractional integral involving fractional exponential function. In fact, our result is a generalization of classical calculus result.**

Keywords: **Jumarie type of R-L fractional calculus, new multiplication, fractional analytic functions, integration by parts for fractional calculus, improper fractional integral, fractional exponential function.**

I. INTRODUCTION

Fractional calculus is a natural extension of the traditional calculus. In fact, since the beginning of the theory of differential and integral calculus, some mathematicians have studied their ideas on the calculation of non-integer order derivatives and integrals. During the 18th and 19th centuries, there were many famous scientists such as Euler, Laplace, Fourier, Abel, Liouville, Grunwald, Letnikov, Riemann, Laurent, Heaviside, and some others who reported interesting results within fractional calculus. In recent years, fractional calculus has become an increasingly popular research area due to its effective applications in different scientific fields such as economics, viscoelasticity, physics, mechanics, biology, electrical engineering, control theory, and so on [1-12].

However, different from the traditional calculus, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivatives. Other useful definitions include Caputo fractional derivatives, Grunwald-Letnikov (G-L) fractional derivatives, and Jumarie type of R-L fractional derivatives to avoid non-zero fractional derivative of constant function [13-17].

In this paper, based on Jumarie's modified R-L fractional calculus and a new multiplication of fractional analytic functions, we use integration by parts for fractional calculus to solve an improper fractional integral involving fractional exponential function. Moreover, our result is a generalization of traditional calculus result.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([18])**:** Let $0 < \alpha \leq 1$, and x_0 be a real number. The Jumarie's modified Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)] = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,, \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

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$$\left({}_{x_0}I_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{2}$$

where $\Gamma(\ \ )$ is the gamma function.

Proposition 2.2 ([19])**:** *If* α,β,x_0,C *are real numbers and* $\beta\geq\alpha>0$, *then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right]=\frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{3}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C]=0. \tag{4}$$

Definition 2.3 ([20])**:** If x,x_0, and a_n are real numbers for all n, $x_0\in(a,b)$, and $0<\alpha\leq 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. In addition, if $f_\alpha:[a,b]\to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([21])**:** If $0<\alpha\leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x=x_0$,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{5}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{6}$$

Then

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}\\&=\sum_{n=0}^{\infty}\frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)(x-x_0)^{n\alpha}.\end{aligned} \tag{7}$$

Equivalently,

$$\begin{aligned}&f_\alpha(x^\alpha)\otimes_\alpha g_\alpha(x^\alpha)\\&=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\otimes_\alpha\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}\\&=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m}a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}.\end{aligned} \tag{8}$$

Definition 2.5 ([22])**:** Let $0<\alpha\leq 1$, and $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ be two α-fractional analytic functions defined on an interval containing x_0 ,

$$f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes n}, \tag{9}$$

$$g_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes n}. \tag{10}$$

The compositions of $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are defined by

$$(f_\alpha\circ g_\alpha)(x^\alpha)=f_\alpha\left(g_\alpha(x^\alpha)\right)=\sum_{n=0}^{\infty}\frac{a_n}{n!}\left(g_\alpha(x^\alpha)\right)^{\otimes n}, \tag{11}$$

and

$$(g_\alpha\circ f_\alpha)(x^\alpha)=g_\alpha\left(f_\alpha(x^\alpha)\right)=\sum_{n=0}^{\infty}\frac{b_n}{n!}\left(f_\alpha(x^\alpha)\right)^{\otimes n}. \tag{12}$$

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Definition 2.6 ([22])**:** Let $0 < \alpha \leq 1$. If $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are two α-fractional analytic functions satisfies

$$(f_\alpha \circ g_\alpha)(x^\alpha) = (g_\alpha \circ f_\alpha)(x^\alpha) = \frac{1}{\Gamma(\alpha+1)} x^\alpha. \tag{13}$$

Then $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ are called inverse functions of each other.

Definition 2.7 ([23])**:** If $0 < \alpha \leq 1$, and x is a real number. The α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha n}. \tag{14}$$

And the α-fractional logarithmic function $Ln_\alpha(x^\alpha)$ is the inverse function of $E_\alpha(x^\alpha)$. On the other hand, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha 2n}, \tag{15}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha\right)^{\otimes_\alpha (2n+1)}. \tag{16}$$

Theorem 2.8 (integration by parts for fractional calculus) ([24])**:** *Assume that* $0 < \alpha \leq 1$, a, b *are real numbers, and* $f_\alpha(x^\alpha)$, $g_\alpha(x^\alpha)$ *are* α*-fractional analytic functions, then*

$$\left({}_aI_b^\alpha\right)\left[f_\alpha(x^\alpha) \otimes_\alpha \left({}_aD_x^\alpha\right)[g_\alpha(x^\alpha)]\right] = \left[f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha)\right]_{x=a}^{x=b} - \left({}_aI_b^\alpha\right)\left[g_\alpha(x^\alpha) \otimes_\alpha \left({}_aD_x^\alpha\right)[f_\alpha(x^\alpha)]\right]. \tag{17}$$

III. MAIN RESULT

In this section, we solve an improper fractional integral involving fractional exponential function.

Theorem 3.1: *Let* $0 < \alpha \leq 1$, *then the improper* α*-fractional integral*

$$\left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes_\alpha E_\alpha(-x^\alpha) \otimes_\alpha [1 + E_\alpha(-x^\alpha)]^{\otimes_\alpha (-2)}\right] = Ln_\alpha(2). \tag{18}$$

Proof $\left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes_\alpha E_\alpha(-x^\alpha) \otimes_\alpha [1 + E_\alpha(-x^\alpha)]^{\otimes_\alpha (-2)}\right]$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes_\alpha E_\alpha(x^\alpha) \otimes_\alpha [1 + E_\alpha(x^\alpha)]^{\otimes_\alpha (-2)}\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes_\alpha \left({}_0D_x^\alpha\right)\left[-[1 + E_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}\right]\right]$$

$$= \left[-\frac{1}{\Gamma(\alpha+1)} x^\alpha \otimes_\alpha [1 + E_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}\right]_{x=0}^{x=+\infty} + \left({}_0I_{+\infty}^\alpha\right)\left[[1 + E_\alpha(x^\alpha)]^{\otimes_\alpha (-1)} \otimes_\alpha \left({}_0D_x^\alpha\right)\left[\frac{1}{\Gamma(\alpha+1)} x^\alpha\right]\right]$$

(by integration by parts for fractional calculus)

$$= 0 - 0 + \left({}_0I_{+\infty}^\alpha\right)\left[[1 + E_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[[1 + E_\alpha(x^\alpha)]^{\otimes_\alpha (-1)}\right]$$

$$= \left({}_0I_{+\infty}^\alpha\right)\left[E_\alpha(-x^\alpha) \otimes_\alpha [1 + E_\alpha(-x^\alpha)]^{\otimes_\alpha (-1)}\right]$$

$$= -\left({}_0I_{+\infty}^\alpha\right)\left[\left({}_0D_x^\alpha\right)[Ln_\alpha(1 + E_\alpha(-x^\alpha))]\right]$$

$$= -\left[Ln_\alpha(1 + E_\alpha(-x^\alpha))\right]_{x=0}^{x=+\infty}$$

$$= Ln_\alpha(2).$$

Q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional calculus and a new multiplication of fractional analytic functions, we can solve an improper fractional integral involving fractional exponential function by using integration by parts for fractional calculus. In addition, our result is a generalization of classical calculus result. In the future, we will continue to use Jumarie type of R-L fractional calculus and the new multiplication of fractional analytic functions to solve problems in fractional differential equations and engineering mathematics.

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